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PROGRESS REPORT.

The following gentlemen have been elected members of the Society :—

1. Pandit Hemraj, M.A.—Professor of Mathematics, Dayal Singh College, Lahore;
2. Mr. A. Narasinga Rao, B.A.—43, Raja Street, Big Conjeeveram (at concessional rate).

2. The committee regret to record the sad death of Mr. C. B. N. Cama, M.A. (Bombay & Cantab), LL.M. (Cantab), I.C.S., and Professor S. P. Singaravelu Mudaliar, B.A., L.T., of Pachaiyappa's College, Madras.

The late Mr. Cama was an enthusiastic member of our Society from its very commencement, a high class mathematician and a man of exceptional ability. Latterly owing to his official duties and his devotion to the study of several Oriental and Indian languages, he was not able to do any serious work in mathematics; yet his instinctive love and partiality for mathematics remained with him up to the last. By his untimely death, we lose a valuable member.

The late Professor Singaravelu Mudaliar was a very able and enthusiastic teacher of mathematics on the Madras side, and his popularity was evinced by the large number of students who generally flocked to his college for admission. He was an unostentatious but a distinguished professor, and his death at the early age of forty-two has deprived our Society of a useful member.

POONA,
31st July 1915.

D. D. KAPADIA,
Hony. Joint Secretary.

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts of the Society for the half year ended 30th June 1915.

Receipts.	Rs.	A.	P.	Expenditure.	Rs.	A.	P.
Opening Balance	508	10	2	Books and Periodicals...	332	10	2
Subscriptions from Members	1,355	11	0	Printing Journal	276	0	6
Subscriptions for Journal	149	0	0	Library	120	0	0
Other receipts	22	11	0	Working Charges	102	3	3
				Closing Balance *	1,205	2	3
Total ...	2,036	0	2	Total ...	2,036	0	2

	Rs.	A.	P.
* With the Treasurer	946	2	0
,, Secretary	28	6	6
,, Assistant Librarian ...	126	0	0
,, ,, Secretary ...	104	9	9
Total ...	1,205	2	3

MADRAS,
30th July 1915.

S. NARAYANA AIYAR,
Hon. Treasurer.

Forces along non-intersecting Generators of a Hyperboloid.

By R. VYTHYNATHASWAMY, B.A.

[In this paper some properties of systems of forces along generators of the same system of a hyperboloid are deduced by combining the theory of the *nul point* and the *nul plane* with that of the *polar point* and the *polar plane*. A brief account of such systems of forces is to be found in Routh's *Analytical Statics*, Vol. I, Chapter VII, to which the reader is referred.]

1. *The identical relation between the mutual moments of seven lines in space.*

If $(l_r, m_r, n_r, \lambda_r, \mu_r, \nu_r)(r=1, 2, \ldots 7)$ be the co-ordinates of the lines and (rs) represent the moment of $(l_r \ldots)$ and $(l_s \ldots)$, it is readily seen by the multiplication of the arrays

$$\| l_r\, m_r\, n_r\, \lambda_r\, \mu_r\, \nu_r \|, \ \| \lambda_r\, \mu_r\, \nu_r\, l_r\, m_r\, n_r \| \qquad (r=1, 2 \ldots 7)$$

that the required relation is

$$\begin{vmatrix} 0 & (12) & (13) & (14) & (15) & (16) & (17) \\ (21) & 0 & (23) & (24) & (25) & (26) & (27) \\ (31) & (32) & 0 & (34) & (35) & (36) & (37) \\ (41) & (42) & (43) & 0 & (45) & (46) & (47) \\ (51) & (52) & (53) & (54) & 0 & (56) & (57) \\ (61) & (62) & (63) & (64) & (65) & 0 & (67) \\ (71) & (72) & (73) & (74) & (75) & (76) & 0 \end{vmatrix} = 0.$$

2. *The necessary and sufficient condition that the skew lines a, b, c, d lie on a hyperboloid is*

$$\Delta \equiv \begin{vmatrix} 0 & (ab) & (ac) & (ad) \\ (ba) & 0 & (bc) & (bd) \\ (ca) & (cb) & 0 & (cd) \\ (da) & (db) & (dc) & 0 \end{vmatrix} = 0$$

or $$p^2+q^2+r^2-2qr-2rp-2pq=0,$$

where $p=(bc)(ad)$, $q=(ca)(bd)$, $r=(ab)(cd)$.

First :—The condition is necessary. For, if a, b, c, d lie on a hyperboloid, then there are three lines e, f, g which meet them all. The identical relation of the previous article becomes

$$\Delta \times \begin{vmatrix} 0 & (ef) & (eg) \\ (fe) & 0 & (fg) \\ (ge) & (gf) & 0 \end{vmatrix} = 0.$$

Since e, f, g are skew, the second determinant cannot vanish and hence

$$\Delta \equiv 0.$$

To prove the second part ; Let e, f be the determinate lines which meet a, b, c, d and g an indeterminate line meeting a, b, c. (Routh, § 320). Since Δ is given to vanish, the fundamental identical relation assumes the form

$$(dg)^2\phi_1 + (dg)\phi_2 = 0,$$

where ϕ_1, ϕ_2 do not depend on g.

Hence (dg) is either zero or constant for all positions of g.

In the latter case also (dg) is zero, since e, f are possible positions of g.

Thus g meets d and hence a, b, c, d lie on a hyperboloid.

3. *If four non-intersecting forces are in equilibrium their lines of action lie on a hyperboloid.* (Routh, § 316.)

Let P, Q, R, S be the forces, a, b, c, d their lines of action.

Taking moments about a, b, c, d and eliminating P, Q, R, S from the resulting equations, we see that the determinant Δ of the last article vanishes. Thus a, b, c, d lie on a hyperboloid.

Cayley's Theorem :—The forces P, Q, R, S are proportional to $A^{\frac{1}{2}}, B^{\frac{1}{2}}, C^{\frac{1}{2}}, D^{\frac{1}{2}}$ where $A = (bc)(cd)(db)$, etc. (Routh, § 316, Ex. 6.)

Let I_{rs} represent the minor of the constituent of the rth row and s^{th} column in the determinant Δ. Then

$$\frac{P}{I_{11}} = \frac{Q}{I_{12}} = \frac{R}{I_{13}} = \frac{S}{I_{14}}.$$

But by Salmon's *Higher Algebra* $I_{11} \cdot I_{22} = I^2_{12}$.

$$\therefore \quad \frac{P^2}{I_{11}} = \frac{Q^2}{I_{22}} = \frac{R^2}{I_{33}} = \frac{S^2}{I_{44}}.$$

Since $I_{11} = A, \ldots$ the result follows.

Cor.—Resolving along the four lines, we get

$$A^{\frac{1}{2}} + B^{\frac{1}{2}} \cos(\hat{ab}) + C^{\frac{1}{2}} \cos(ac) + D^{\frac{1}{2}} \cos(\hat{ad}) \equiv 0,$$

and three similar relations.

It is easily seen, that the forces are also proportional to the lines of the solid angles between every three of the four lines.

4. *Nul Lines and Planes* :

Consider a system of forces along generators of the λ-system of a hyperboloid.

(i) Every μ—generator is a *nul line* of the system since it meets all the forces.

(ii) The nul plane of any point A on the hyperboloid passes through the μ—generator through A and is therefore the tangent plane to the hyperboloid at some point B of this generator.

(iii) The system can be reduced to two conjugate forces along generators of the λ—system. (Routh, § 303.)

For the conjugate of a λ—generator must meet all the nul lines which meet that generator. That is, it must meet all the μ—generators and is consequently a λ—generator.

Def. Two such generators shall be called *conjugate generators* of the system.

5. *If the nul plane of A is the tangent plane at B, then the nul plane of B is the tangent plane at A.*

Let a, a' be the λ—and μ—generators through A, and let b be the generator conjugate to a. Then the nul plane of A passes through b and is the plane (Ab) or ($a'b$) and is thus the tangent plane at the intersection of a' and b. Hence B is the intersection of a' and b.

Similarly, the nul plane of B is the plane (Ba) or (aa') and is therefore the tangent plane at A.

Def :—Two points on the hyperboloid may be said to be *conjugate,* when the nul plane of each is the tangent plane at the other. Thus it appears from the above that any two conjugate generators cut any μ-generator in conjugate points.

6. *If P,P′ be the ends of a diameter of the hyperboloid and Q,Q′ their conjugates, then QQ′ will be parallel to the principal force of the system.*

For the nul planes of Q and Q′ must be parallel, which means that the tangent planes at the conjugates P, P′ should be parallel. Hence the result.

7. *Self-conjugate points and self-conjugate generators* :—If QQ′ be itself a diameter, then PP′ and QQ′ will coincide. In other words, the principal force at the centre meets the hyperboloid in *self-conjugate points.* The λ—generator of either of these self-conjugate points is a *self-conjugate generator* ; every point of a self-conjugate generator is a self-conjugate point, since conjugate generators must pass through

conjugate points. The *self-conjugate generators* are nul lines of the system, since they lie in the nul planes of the self-conjugate points.

There is some difficulty in the proper conception of a self-conjugate generator. A self-conjugate generator does not mean that the given system of forces reduces to a single force along it, as may appear at first sight. When two conjugate generators coincide what happens really is that the equivalent forces along these become infinite and by pulling in opposite directions along the self-conjugate produce the same effect as a finite force and a couple. (See Routh, § 306.)

8. *The nul plane at the centre is the diametral plane of the principal force.* (Routh, § 317, Ex. 9.)

For the nul plane at the centre is parallel to the nul plane of any other point on the principal force at the centre and is thus parallel to the nul plane at the self-conjugate point in which the principal force meets the hyperboloid. Now, this latter is the tangent plane at the self-conjugate point, and therefore the parallel central plane has the property stated.

9. The result of § 6 may be generalized as follows:—*If PP' be a chord of the hyperboloid passing through a fixed point L, and (Q, Q') the conjugates of (P, P'), then the chord QQ' passes through another fixed point, M, which is the pole of the nul plane of L.*

For, the nul planes of points on a line have a common line of intersection (Routh § 303.) Therefore the nul planes of P, P′ intersect in the nul plane of L. But the nul planes of P, P′ are the tangent planes at Q, Q′. Hence the tangent planes at Q, Q′ intersect in the nul plane of L. By reciprocation, the chord QQ′ passes through the pole of the nul plane of L, which is a fixed point.

Cor. Let R be the principal force at the centre and R_1, R_2 two conjugate diameters in the diametral plane of R; then if PP′ be a chord parallel to R_1, QQ′ will be parallel to R_2. For, in this case L may be regarded as the point at infinity on R_1, whose nul plane is the plane (RR_1); and therefore QQ′ always passes through the pole of the plane (RR_1), *viz*: the point at infinity on R_2, that is QQ′ is always parallel to R_2.

10. *Cross ratio*:—The cross ratio of four points on a line is equal to the cross ratio of their nul planes with respect to any set of forces. For, the nul planes of the four points pass through the same conjugate line and are cut by the given line at these points. The following involution properties are noteworthy:

(i) *Pairs of conjugate generators are in involution, that is, they cut any μ—generator in a range in involution.*

Let pp', qq', rr', be pairs of conjugate generators and let a μ-generator cut them in PP′, QQ′, RR′, which are pairs of conjugate points. Then

$$\begin{aligned}(PQRP') &= \text{cross ratio of nul planes of } PQRP' \\ &= \text{cross ratio of tangent planes at } P'Q'R'P \\ &= (P'Q'R'P.)\end{aligned}$$

Thus PP′, QQ′, RR′ belong to the same involution.

Hence pp', qq', rr' are in involution.

(ii) *The two self-conjugate generators and any two conjugate generators form a harmonic set; that is, they intercept on any μ-generator a harmonic range.*

For, the self-conjugates are the double lines of the involution to which pairs of conjugate generators belong.

(iii) *Pairs of conjugate points on a μ-generator belong to an involution of which the self-conjugate points are the double points.*

N.B.—The truth of the above theorems, may also be inferred by conceiving the hyperboloid as a one-dimensional space the elements of which are λ-generators, in as much as the correspondence of conjugacy is reciprocal.

11. *The central axis:—The central axis of the system is the shortest distance between the two μ-generators which are perpendicular to the principal force.*

Let the central axis intersect the hyperboloid in P, Q and let p, q be the μ-generators through P, Q.

Since the nul planes of P, Q are perpendicular to the central axis, the lines p, q which lie in these planes are also perpendicular to the central axis, and therefore to the principal force.

Hence the central axis is the shortest distance between p and q.

Some further Generalizations :

12. *If L, M be any two points (not necessarily on the hyperboloid) such that the nul plane of L is the polar plane of M, then the nul plane of M is the polar plane of L.*

For, it has been proved in § 9 that, if PP′ be any chord through L, and (Q, Q′) the conjugates of (P, P′) then the chord QQ′ passes through M the pole of the nul plane of L. Since the relations of L and M are reciprocal, the result stated follows at once.

Def. Two points L, M such that the nul plane of each is the polar plane of the other may be said to be *conjugate* with respect to our system of forces. Also two lines such as PP′, QQ′ may be called *corresponding lines* of the system.

[*Note.* This definition of conjugate points does not conflict with the definition in § 5 above. For, if L is on the hyperboloid, M will be its conjugate as previously defined.]

13. *Two corresponding lines are such that the 'conjugate line' of either is the 'polar line' of the other.**

Let PP′, QQ′ be the corresponding lines and l, m, their conjugate lines. Then, the nul plane of any point L on PP′ passes through l and is the polar plane of a point M on QQ′ (§ 12); and the polar line of QQ′ is the common section of the polar planes of different points on it. Hence l is the polar line of QQ′.

Cor. 1. Conjugate λ—generators are corresponding lines of the system.

Cor. 2. The locus of points conjugate to points on a given line is the corresponding line.

Cor. 3. To find the line corresponding to a given line: let the given line cut the hyperboloid in P, P′; join their conjugate points Q Q′; QQ′ is the corresponding line required.

Cor. 4. If PP′ and QQ′ coincide, we have *a self-corresponding line.* This may happen in either of two ways:

(i) P may coincide with Q and P′ with Q′; in which case the self-corresponding line is a line joining any two *self-conjugate* points.

This includes the case of *self-conjugate generators* also.

(ii) P may coincide with Q′ and P′ with Q, in which case the self-corresponding line is the line joining *any point* on the hyperboloid to *its conjugate* and is a μ—generator.

Hence a self-corresponding line *generally* meets the self-conjugate generators.

14. *The line joining any point and its conjugate (not necessarily on the hyperboloid) is a self-corresponding line.*

Let L, M be a pair of conjugate points. Then the polar line of LM is the intersection of the polar planes of L and M, which are the nul planes of M and L respectively. Thus the polar line of LM is identical with the conjugate line. Hence LM is a self-corresponding line.

15. *A pair of corresponding lines and the self-conjugate generators form generators of the same system of another hyperboloid.*

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* The expression 'polar line' is here used to denote the common line of intersection of the *polar planes* of points on a given line, to distinguish it from the line of action of the conjugate force which is here called 'the conjugate line'.

For, the join of a pair of conjugate points, one on each of the corresponding lines is a self-corresponding line and therefore meets the self-conjugate generators. Hence, the four lines in question form generators of the same system of a certain hyperboloid and the joining line is a generator of the opposite system.

16. Consider a hyperboloid formed as above, with the self-conjugates and a given pair of corresponding lines as its λ—generators. It is evident that the corresponding line of any λ—generator of this hyperboloid will be another λ—generator.

To prove that pairs of such corresponding λ—generators are in involution.

Let pp', qq', rr' be three such pairs of corresponding λ—generators. Let them cut a self-corresponding line in PP′, QQ′, RR′. These form pairs of conjugate points and we have

(PQRP′) = cross ratio of nul planes of PQRP′,
= cross ratio of polar planes of P′Q′R′P,
= (P′Q′R′P.)

Hence (PP′, QQ′, RR′) and therefore (pp', qq', rr') are in involution.

The self-conjugates are the double lines of this involution, and we may therefore say that a pair of corresponding lines and the self-conjugate generators form a *harmonic* system as in § 10.

Similarly, pairs of conjugate points on a self-corresponding line belong to an involution the double points of which are the self-conjugate points on the line.

17. *If (pp', qq') are two pairs of skew lines, there is a unique pair of lines (rr') which separates both of them harmonically.*

For, since the hyperboloids $(pp'rr')$, $(qq'rr')$ have two generators of one system common they have also two generators of the opposite system common. These are the two lines which meet (pp', qq').

Hence the following construction for (rr'). (Routh, § 320.) Let the two lines meeting (pp', qq') cut them in (P_1P_1', Q_1Q_1') (P_2P_2', Q_2Q_2') respectively.

Then rr' are the lines which divide these sets of points harmonically.

Cor.—Two systems of forces acting along λ—generators have in general only one common conjugate line-pair.

For, the only pair of corresponding lines common to both systems is the line-pair separating harmonically the self-conjugates of both systems. There is only one such pair and it lies on the hyperboloid, as shown above. Hence the only pair of common corresponding lines is the pair of conjugate generators separating both the self-conjugates harmonically.

18. We shall conclude with a few properties of the general system of forces :

(i) *Any two pairs of conjugate lines with respect to a general system of forces are generators of the same system of a certain hyperboloid.*

Let (PP′), (QQ′) be the pairs of conjugate forces ; then the forces P, P′, −Q, −Q′ acting along the same lines are in equilibrium. Hence, these lines of action lie on a hyperboloid. (Routh, § 316.)

(ii) *Any two systems of forces have one conjugate line-pair in common.* (Routh, § 311, Ex.)

Through any point there passes a common nul line to both systems, *viz.*, the line of intersection of the nul planes of that point with respect to the two systems. Describe a hyperboloid with three such common nul lines for μ−generators. The given systems of forces can now be reduced to forces along λ−generators of this hyperboloid. Hence the theorem reduces to what has been proved in § 17, *Cor.* above.

(iii) *The common nul lines of three systems of forces generate a hyperboloid.*

Let A, B, C be the three systems ; (pp'), (qq') the common conjugate pairs of (A and B) and (B and C) respectively. Then (pp', qq') lie on a hyperboloid by (i). The generators of the opposite system are the common nul lines of the three systems.

(iv) *Four systems of forces have in general two nul lines in common.*

For, if D be the fourth system and (rr') the common conjugate pair of C and D, the hyperboloids $(pp'qq')$, $(qq'rr')$, have two generators of one system common. Therefore they have two generators of the other system in common. These two meet (pp', qq', rr') and are the common nul lines required.

On the Number of Divisors of a Number.

BY S. RAMANUJAN.

1. If δ be a divisor of N, then there is a conjugate divisor δ' such that $\delta\delta'=N$. Thus we see that

(1) the number of divisors from 1 to $\sqrt{N}$ is equal to the number of divisors from $\sqrt{N}$ to N.

From this it evidently follows that

$$(2) \qquad d\,(N) < 2\sqrt{N},$$

where d (N) denotes the number of divisors of N (including unity and the number itself). This is only a trivial result as all the numbers from 1 to $\sqrt{N}$ cannot be divisors of N. So let us try to find the best possible superior limit for $d(N)$ by using purely elementary reasoning.

2. First let us consider the case in which all the prime divisors of N are known. Let

$$N = p_1^{a_1} . p_2^{a_2} . p_3^{a_3} . \ldots \qquad \ldots p_n^{a_n},$$

where $p_1, p_2, p_3 \ldots p_n$ are a given set of n primes. Then it is easy to see that

$$(3) \qquad d\,(N) = (1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n).$$

But

$$(4) \quad \frac{1}{n}\left\{(1+a_1)\log p_1 + (1+a_2)\log p_2 + \ldots + (1+a_n)\log p_n\right\}$$
$$> \left\{(1+a_1)(1+a_2)\ldots(1+a_n)\log p_1 \log p_2 \ldots \log p_n\right\}^{1/n}$$

since the arithmetic mean of unequal positive numbers is always greater than their geometric mean. Hence

$$\frac{1}{n}\left\{\log p_1 + \log p_2 + \ldots + \log p_n + \log N\right\}$$
$$> \left\{\log p_1 \log p_2 \ldots \log p_n . d(N)\right\}^{1/n}$$

In other words

$$(5) \qquad d(N) < \frac{\frac{1}{n}\left\{\log\,(p_1 p_2 p_3 \ldots\ldots p_n N)\right\}^n}{\log p_1 \log p_2 \log p_3 \ldots\ldots \log p_n},$$

for all values of N whose prime divisors are $p_1, p_2, p_3 \ldots\ldots p_n$.

3. Next let us consider the case in which only the number of prime divisors of N is known. Let

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots \qquad \ldots \qquad \ldots p_n^{a_n},$$

where n is a given number; and let

$$N' = 2^{a_1} . 3^{a_2} . 5^{a_3} \ldots \qquad \ldots \qquad \ldots p^{a_n},$$

where p is the natural nth prime. Then it is evident that

(6) $$N' \leq N;$$

and (7) $$d(N') = d(N).$$

But

(8) $$d(N') < \frac{\left\{\frac{1}{n}\log(2.3.5\ldots\ldots p.N')\right\}^n}{\log 2 \log 3 \log 5 \ldots\ldots \log p}$$

by virtue of (5). It follows from (6) to (8) that, if p be the natural nth prime, then

(9) $$d(N) < \frac{\left\{\frac{1}{n}\log(2.3.5\ldots\ldots p.N)\right\}^n}{\log 2 \log 3 \log 5 \ldots \log p},$$

for all values of N having n prime divisors.

4. Finally, let us consider the case in which nothing is known about N. Any integer N can be written in the form

$$2^{a_2}.3^{a_3}.5^{a_5}\ldots\ldots\ldots,$$

where $a_\lambda \geq 0$. Now let

(10) $$x^h = 2,$$

where h is any positive number. Then we have

(11) $$\frac{d(N)}{N^h} = \frac{1+a_2}{2^{ha_2}}.\frac{1+a_3}{3^{ha_3}}.\frac{1+a_5}{5^{ha_5}}\ldots\ldots\ldots$$

But from (10) we see that, if q be any prime greater than x, then

(12) $$\frac{1+a_q}{q^{ha_q}} \leq \frac{1+a_q}{x^{ha_q}} = \frac{1+a_q}{2^{a_q}} \leq 1.$$

It follows from (11) and (12) that, if p be the largest prime not exceeding x, then

(13) $$\frac{d(N)}{N^h} \leq \frac{1+a_2}{2^{ha_2}}.\frac{1+a_3}{3^{ha_3}}.\frac{1+a_5}{5^{ha_5}}\ldots\ldots\ldots\frac{1+a_p}{p^{ha_p}}.$$
$$\leq \frac{1+a_2}{2^{ha_2}}.\frac{1+a_3}{2^{ha_3}}.\frac{1+a_5}{2^{ha_5}}\ldots\ldots\ldots\frac{1+a_p}{2^{ha_p}}.$$

But it is to easy to show that the maximum value of $(1+a)\,2^{-ha}$ for the variable a is $\frac{2^h}{(h\,e\log 2)}$. Hence,

(14) $$\frac{d(N)}{N^h} \leq \left(\frac{2^h}{h\,e\log 2}\right)^{\omega(x)}$$

where $\omega(x)$ denotes the number of primes not exceeding x. But from (10) we have

$$h = \frac{\log 2}{\log x}.$$

Substituting this in (14), we obtain

$$(15)\qquad d(\mathrm{N}) \leq \mathrm{N}^{\frac{\log 2}{\log x}} \cdot \left\{ \frac{2^{\frac{\log 2}{\log x}} \log x}{e\,(\log 2)^2} \right\}^{\omega(x)}.$$

But it is easy to verify that, if $x \geq 6{\cdot}05$, then

$$2^h < e(\log 2)^2.$$

From this and (15) it follows that, if $x \geq 6{\cdot}05$, then

$$(16)\qquad d(\mathrm{N}) < 2^{(\log \mathrm{N})/(\log x)} \, (\log x)^{\omega(x)}$$

for all values of N, $\omega(x)$ being the number of primes not exceeding x.

5. The symbol "O" is used in the following sense:

$$\phi(x) = \mathrm{O}\,\{\psi(x)\}$$

means that there is a positive constant K such that

$$\left|\frac{\phi(x)}{\psi(x)}\right| \leq \mathrm{K}$$

for all sufficiently large values of x. [See Hardy, *Orders of Infinity*, pp. 5 *et seq.*] For example:

$5x = \mathrm{O}(x)$; $\frac{1}{2}x = \mathrm{O}(x)$; $x \sin x = \mathrm{O}(x)$; $\sqrt{x} = \mathrm{O}(x)$; $\log x = \mathrm{O}(x)$;

but $x^2 \neq \mathrm{O}(x)$; $x \log x \neq \mathrm{O}(x)$.

Hence it is obvious that

$$(17)\qquad \omega(x) = \mathrm{O}(x).$$

Now, let us suppose that

$$x = \frac{\log \mathrm{N}}{(\log \log \mathrm{N})^2}$$

in (16). Then we have

$$\log x = \log \log \mathrm{N} + \mathrm{O}(\log \log \log \mathrm{N})\,;$$

and so

$$(18)\qquad \frac{\log \mathrm{N}}{\log x} = \frac{\log \mathrm{N}}{\log \log \mathrm{N}} + \mathrm{O}\left\{\frac{\log \mathrm{N} \log \log \log \mathrm{N}}{(\log \log \mathrm{N})^2}\right\}.$$

Again

$$(19)\qquad \omega(x) \log \log x = \mathrm{O}(x \log \log x)$$

$$= \mathrm{O}\left\{\frac{\log \mathrm{N} \log \log \log \mathrm{N}}{(\log \log \mathrm{N})^2}\right\}.$$

It follows from (16), (18) and (19), that

$$(20)\qquad d(\mathrm{N}) < 2^{(\log \mathrm{N})/\log \log \mathrm{N} + \mathrm{O}\left\{\frac{\log \mathrm{N} \log \log \log \mathrm{N}}{(\log \log \mathrm{N})^2}\right\}}$$

for all sufficiently large values of N.

SHORT NOTES.

The Axes of a Conicoid and an Allied Cone.

The following method of determining the axes of the general conicoid may be of interest as it is an extension of the method usually adopted in the case of the general conic

1. Let $\phi(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$ be the conicoid referred to rectangular axes through the centre. Then a concentric sphere of radius r will cut the conicoid in sphero-conics lying on the cone

$$\phi-(x^2+y^2+z^2)/r^2=0 \quad \ldots \quad \ldots \quad \ldots (1)$$

When the sphere touches the conicoid, r will be equal to one of the principal semi-axes, and the cone (1) will in this case denote the line of contact of the sphere and the conicoid, that is, one of the principal axes.

The condition for this is obtained by identifying (1) with

$$\alpha\left(\frac{y}{m}-\frac{z}{n}\right)^2+\beta\left(\frac{z}{n}-\frac{x}{l}\right)^2+\gamma\left(\frac{x}{l}-\frac{y}{m}\right)^2=0 \ldots \quad \ldots (2)$$

where l, m, n are the direction cosines of the line of contact and α, β γ are unknown. Thus, we may write

$$a-1/r^2=(\beta+\gamma)/l^2, \ldots\ldots$$
$$f=-\alpha/mn, \ldots\ldots\ldots$$

whence we infer

$$\left.\begin{aligned}(a-\lambda)\,l+h\,m+g\,n&=0,\\ h\,l+(b-\lambda)\,m+fn&=0,\\ g\,l+f\,m+(c-\lambda)\,n&=0,\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots (3)$$

putting $\lambda=1/r^2$.

Eliminating l, m, n we get the usual determinant

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix}=0. \quad \ldots \quad \ldots (4)$$

2. Equation (4) can also be derived as the condition that (1) should represent two central planes, which must be the circular sections of the conicoid.

The circular sections are therefore written

$$(a-\lambda)\,x+hy+gz=\pm\left[\{h^2-(a-\lambda)(b-\lambda)\}\,y^2 + 2\{gh-f(a-\lambda)\}\,yz+\{g^2-(a-\lambda)(c-\lambda)\}\,z^2\right]^{\frac{1}{2}}$$

where the expression within the square brackets is a perfect square.

∴ In other words, the circular sections are

$$(a-\lambda)\,x+hy+gz=\pm\,\frac{\{[h^2-(a-\lambda)\,(b-\lambda)]\,y+[gh-f\,(a-\lambda)z]\}}{\sqrt{h^2-(a-\lambda)\,(b-\lambda)}}$$

3. For all values of r the cone (1) has its principal axes in the same direction as the conicoid ϕ. For, the discriminating cubic of the cone differs from that of the conicoid in having

$$\left(a-\frac{1}{r^2}\right),\ \left(b-\frac{1}{r^2}\right),\ \left(c-\frac{1}{r^2}\right),$$

in place of a, b, c; so that the roots of the former must be

$$\left(\lambda_1-\frac{1}{r^2}\right),\ \left(\lambda_2-\frac{1}{r^2}\right),\ \left(\lambda_3-\frac{1}{r^2}\right)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the latter. Hence, the equations for the direction cosines of the axes of the conic will remain the same as (3).

M. T. NARANIENGAR.

The four-point Conic of minimum Eccentricity.

1. Let P, Q, R, S be four given points. Then, we know that 'the axes of the two parabolas through the points are parallel to conjugate diameters of any conic through the same.' [Smith's *Conic Sections*, 1910; p. 292.] Now, in a given ellipse the acute angle between two conjugate diameters is least when they are equi-conjugates, and in different ellipses the angle between the equi-conjugates is greatest when the eccentricity is least. Thus, if the angle between a pair of conjugate diameters is known, the conic will have a minimum eccentricity, when these diameters are the equi-conjugates.

Hence, we have the following: *The conic of minimum eccentricity through four given points has its equi-conjugates parallel to the axes of the two parabolas through the points.*

2. Again 'if the normals to an ellipse at four points are concurrent, the axis of the parabola through the points is parallel to one or other of the equi-conjugates.' (Ex. 3, p. 269). Hence we infer that *this conic of minimum eccentricity is such that the normals at the points P, Q, R, S, are concurrent.*

3. Lastly 'if normals at P, Q, R, S are concurrent and the circle PQR meet the conic in S′, then SS′ is a diameter' (Ex. 2, p. 269). Hence the centre of the conic is the middle point of SS.′

If p, q, r be the mid-points of SP, SQ, SR and p', q', r' of QR, RP, PQ this centre lies on the circle pqr and is obviously the common

point of the four circles pqr, $pq'r'$, $p'qr'$, $p'q'r$. Also, the centre of the rectangular hyperbola through P, Q, R, S is the common point of the circles $p'q'r'$, $p'qr$, $pq'r$, pqr'.

Hence, *the centre of the ellipse in question and the centre of the rectangular hyperbola are equidistant from and collinear with the mean centre of the four points.*

N. DURAI RAJAN.

Notes on Questions 546, 572, 573, 583, 585, 606.

[Most of the results in these questions follow from some formulæ given (very inaccurately) in CARR's *Synopsis.*]

1. Let a be a positive quantity < 1, so that $\frac{1}{1+a}$ and $\frac{1-a}{1+a}$ are also positive proper fractions. Then $\phi(a) \equiv \sum_1^\infty (a^n \div n^2)$ is convergent; and it is seen that

$$\phi(a) = -\int_0^a \frac{\log(1-x)}{x}dx,\ \phi(-a) = -\int_0^a \frac{\log(1+x)}{x}dx;$$

$$\phi(1) = \frac{\pi^2}{6},\ \phi(-1) = -\frac{\pi^2}{12};\ \phi(a)+\phi(-a) = \tfrac{1}{2}\phi(a^2),$$

and $$\phi(a)-\phi(-a) = \int_0^a \frac{1}{x}\log\frac{1+x}{1-x}dx = 2\int_0^a \frac{\tanh^{-1}x}{x}dx.$$

We shall use $f(a)$ for

$$\int_0^a \frac{\tanh^{-1}x}{x}dx \equiv \sum_0^\infty \left[a^{2n+1} \div (2n+1)^2\right];$$

thus $f(1)$ is $\frac{\pi^2}{8}$.

Differentiating $\log x \log(1-x)$, we get at once

$$\Big\{\log x \log(1-x)\Big\}_0^a = -\int_0^a \frac{\log x}{1-x}dx + \int_0^a \frac{\log(1-x)}{x}dx$$

$$= \int_1^{1-a} \frac{\log(1-x)}{x}dx + \int_0^a \frac{\log(1-x)}{x}dx$$

$$= -\phi(1-a)+\phi(1)-\phi(a);$$

so that $$\phi(a)+\phi(1-a) = \frac{\pi^2}{6} - \log a \log(1-a); \qquad \text{... (A)}$$

Similarly $\left\{\log x \log (1+x) - \frac{1}{2} [\log (1+x)]^2\right\}_0^a$

$$= \int_0^a \frac{\log x}{1+x} dx + \int_0^a \frac{\log (1+x)}{x} dx - \int_0^a \frac{\log (1+x)}{1+x} dx$$

$$= -\phi(-a) + \int_0^a \frac{1}{1+x} \log \frac{x}{1+x} dx$$

$$= -\phi(-a) - \int_1^{\frac{1}{1+a}} \frac{\log (1-x)}{x} dx,$$

$$= -\phi(-a) + \phi\left(\frac{1}{1+a}\right) - \frac{\pi^2}{6};$$

so that $\phi\left(\frac{1}{1+a}\right) - \phi(-a) = \frac{\pi^2}{6} + \log a \log (1+a) - \frac{1}{2} [\log (1+a)]^2$.

The left side can also be written

$$\phi(a) - \tfrac{1}{2}\phi(a^2) + \phi\left(\frac{1}{1+a}\right),\ 2f(a) - \phi(a) + \phi\left(\frac{1}{1+a}\right) \quad \ldots\ldots \text{(B)}$$

Again, differentiating $\log x \tanh^{-1} x$, we get

$$\log a \tanh^{-1} a = \int_c^a \frac{\tanh^{-1} x}{x} dx + \int^a \frac{\log x}{1-x^2} dx.$$

Changing x to $\frac{1-x}{1+x}$, we have

$$\int_0^a \frac{\log x\, dx}{1-x^2} = \int_1^{\frac{1-a}{1+a}} \frac{\tanh^{-1} x}{x} dx = f\left(\frac{1-a}{1+a}\right) - f(1);$$

hence $\quad f(a) + f\left(\frac{1-a}{1+a}\right) = \frac{\pi^2}{8} + \frac{1}{2} \log a \log \left(\frac{1+a}{1-a}\right) \quad \ldots \text{(C)}$

When $a = \frac{1}{2} = 1-a$, (A) gives

$$2\phi(\tfrac{1}{2}) = \frac{\pi^2}{6} - (\log \tfrac{1}{2})^2,$$

or

$$\sum_1^\infty \frac{1}{2^n n^2} = \frac{\pi^2}{12} - \tfrac{1}{2} (\log 2)^2.$$

When $a = \frac{1-a}{1+a}$, so that a is the positive fraction $\sqrt{2}-1$, (C) gives

$$2f(\sqrt{2}-1) = \frac{\pi^2}{8} - \tfrac{1}{2}\{\log (\sqrt{2}-1)\}^2;$$

this is result (i) in Question 572, which will be found to be identical with result (ii) in Q. 546.

When $a=\frac{1}{1+a}$, so that $a^2+a=1$ and a is the positive fraction $\frac{1}{2}(\sqrt{5}-1)$, (B) gives

$$2\phi\left(\frac{\sqrt{5}-1}{2}\right)-\tfrac{1}{2}\phi\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{6}-\tfrac{3}{2}(\log a)^2;$$

and (A) gives

$$\phi\left(\frac{\sqrt{5}-1}{2}\right)+\phi\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{6}-2(\log a)^2;$$

solving these equations, we get

$$\phi\left(\frac{\sqrt{5}-1}{2}\right) \text{ or } \sum_1^\infty \frac{1}{n^2}\left(\frac{\sqrt{5}-1}{2}\right)^n=\frac{\pi^2}{10}-\left(\log\frac{\sqrt{5}-1}{2}\right)^2,$$

and $$\phi\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{15}-\left(\log\frac{\sqrt{5}-1}{2}\right)^2.$$

The first of these is the second result in Question 583.

The last formula in (B) at once gives in this case

$$2f(a)=\frac{\pi^2}{6}-\frac{3}{2}(\log a)^2,$$

i.e., $$f\left(\frac{\sqrt{5}-1}{2}\right)=\frac{\pi^2}{12}-\frac{3}{4}\left(\log\frac{\sqrt{5}-1}{2}\right)^2.$$

This is result (ii) in uestion 572, where there is an evident mistake.

Since $$\phi\left(-\frac{\sqrt{5}-1}{2}\right)=\tfrac{1}{2}\phi\left(\frac{3-\sqrt{5}}{2}\right)-\phi\left(\frac{\sqrt{5}-1}{2}\right),$$

therefore $$-\int_1^{\frac{\sqrt{5}-1}{2}}\frac{\log(1+x)}{x}dx=\tfrac{1}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2-\frac{\pi^2}{15},$$

which is the first result in Question 583.

Putting $a=\frac{\sqrt{5}-1}{2}$ and remembering that $\frac{1-a}{1+a}=a^3=\sqrt{5}-2$, we see that we have from (C)

$$f\left(\frac{\sqrt{5}-1}{2}\right)+f(\sqrt{5}-2)=\frac{\pi^2}{8}-\tfrac{1}{2}\log a\log a^3$$

$$=\frac{\pi^2}{8}-\tfrac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2;$$

so that $$f(\sqrt{5}-2)=\frac{\pi^2}{24}-\tfrac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2.$$

This may be put into the form

$$\frac{\pi^2}{24}-\tfrac{3}{4}\left\{\tfrac{1}{3}\left(\log\frac{\sqrt{5}-1}{2}\right)^3\right\}^2=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log(\sqrt{5}-2)\right\}^2;$$

these are the results in Question 572 (iii) and Q. 606, which are thus found to be identical.

3. We have $\int_0^{\frac{\pi}{2}} \tan^{-1}(a \sin \phi)\, d\phi$

$$=a\int_0^{\frac{\pi}{2}} \sin \phi \, d\phi - \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \sin^3 \phi \, d\phi + \frac{a^5}{5}\int_0^{\frac{\pi}{2}} \sin^5 \phi \, d\phi - \ldots$$

$$= a - \frac{2}{3}\cdot\frac{a^3}{3} + \frac{2\cdot 4}{3\cdot 5}\frac{a^5}{5} - \ldots$$

$$= \int_0^a \left(1 - \frac{2}{3}x^2 + \frac{2\cdot 4}{3\cdot 5}x^4 - \ldots\right) dx = \int_0^a \frac{\sinh^{-1} x}{x\sqrt{(1+x^2)}} dx,$$

by a well-known formula.

Put $x = \frac{2z}{1-z^2}$, so that $\sqrt{1+x^2} = \frac{1+z^2}{1-z^2}$, and $\frac{dx}{dz} = \frac{2(1+z^2)}{(1-z^2)^2}$;

also $z = \frac{1}{x}(\sqrt{1+x^2}-1)$, so that the limits become $\frac{1}{a}(\sqrt{1+a^2}-1)$ and 0.

If now $m = \sqrt{1+a^2} - a$,

$$\frac{1}{m} = \sqrt{1+a^2} + a\ ,\quad \sqrt{1+a^2} = \frac{1+m^2}{2m},\ \text{and}\ a = \frac{1-m^2}{2m},$$

so that the upper limit becomes $\frac{1-m}{1+m}$.

Thus the integral is transformed into

$$\int_0^{\frac{1-m}{1+m}} \frac{1}{z} \sinh^{-1}\left(\frac{2z}{1-z^2}\right) dz,$$

i.e.,

$$\int_0^{\frac{1-m}{1+m}} \frac{2\tanh^{-1} z}{z} dz, \text{ or } 2f\left(\frac{1-m}{1+m}\right).$$

Hence, by (C),

$$\int_0^{\frac{\pi}{2}} \tan^{-1}(a \sin \phi)\, d\phi = 2\left(\frac{\pi^2}{8} + \tfrac{1}{2}\log m \log \frac{1+m}{1-m}\right) - 2f(m)$$

$$= \frac{\pi^2}{4} - \log m \log \frac{1-m}{1+m} - 2\sum_0^\infty m^{2n+1} \div (2n+1)^2.$$

This is the result in Question 585, where there is a slight misprint. When $a=1$, $m=\sqrt{2}-1$; and we see that

$$\int_0^{\frac{\pi}{2}} \tan^{-1}(\sin\ \phi)\ d\phi=1-\frac{2}{3^2}+\frac{2\cdot 4}{3\cdot 5^2}-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7^2}+\ldots$$

$$=2f(\sqrt{2}-1)=\frac{\pi^2}{8}-\tfrac{1}{2}\left\{\log(\sqrt{2}-1)\right\}^2,$$

as already deduced above. This is result (ii) in Question 546.

When $a=\frac{1}{2}$, $m=\frac{\sqrt{5}-1}{2}$; and we get

$$\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot 4}{3\cdot 5^2}\cdot\frac{1}{2^5}-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7^2}\cdot\frac{1}{2^7}+\ldots$$

$$=2f(\sqrt{5}-2)=\frac{\pi^2}{12}-\frac{3}{2}\left\{\log\left(\frac{\sqrt{5}-1}{2}\right)\right\}^2.$$

Now the series

$$\frac{1}{3}-\frac{1}{4}-\frac{2}{3^2}\left(\frac{1}{3}-\frac{1}{4^2}\right)+\frac{2\cdot 4}{3\cdot 5^2}\left(\frac{1}{3}-\frac{1}{4^3}\right)-\ldots\ldots$$

$$=\frac{1}{3}\left\{1-\frac{2}{3^2}+\frac{2\cdot 4}{3\cdot 5^2}-\ldots\right\}-\frac{1}{2}\left\{\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot 4}{3\cdot 5^2}\cdot\frac{1}{2^5}-\ldots\ldots\right\}$$

$$=\frac{2}{3}f(\sqrt{2}-1)-f(\sqrt{5}-2)$$

$$=\frac{3}{4}\left\{\log\left(\frac{\sqrt{5}-1}{2}\right)^2-\frac{1}{6}\left\{(\log(\sqrt{2}-1)\right\}^2,\right.$$

as just proved. This result is analogous to the first result in Question 546.

4. If $S(a)$ is put for the series

$$a+\frac{2}{3}\ \frac{a^3}{3}+\frac{2\cdot 4}{3\cdot 5}\ \frac{a^5}{5}+\ldots\ldots\ldots ad.\ inf.,$$

which is the expansion of $\int_0^{\frac{\pi}{2}} \tanh^{-1}(a\tan\phi)d\phi$, we have

$$S(a)=\int_0^a \frac{\sin^{-1}x}{x\sqrt{(1-x^2)}}dx.$$

As in the last paragraph, if $x=\frac{2z}{1+z^2}$, this reduces to $\int_0^b \frac{2\tan^{-1}z}{z}d$, where b is the value of z corresponding to the value a of x, *i.e.*, the positive proper fraction $\{1-\sqrt{(1-a^2)}\}/a$.

If $\int_0^b \frac{\tan^{-1}z}{z}dz$ is denoted by $g(b)$, we have

$$g(1)=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots;$$

also $$g(b)=\log b\ \tan^{-1}b-\int_0^b \frac{\log x}{1+x^2}dx$$

$$=\log b\ \tan^{-1}b-\tfrac{1}{2}\int_0^{2\tan^{-1}b} \log(\tan\tfrac{1}{2}\theta)d\theta,\ \{x=\tan\tfrac{1}{2}\theta\}$$

$$=\log b\ \tan^{-1}b+\int_0^{2\tan^{-1}b}\left(\cos\theta+\frac{\cos 3\theta}{3}+\frac{\cos 5\beta}{5}+\ldots\right)d\theta$$

$$=\log b\ \tan^{-1}b+\frac{\sin\beta}{1^2}+\frac{\sin 3\beta}{3^2}+\frac{\sin 5\beta}{5^2}+\ldots\ldots,$$

where $\beta=2\tan^{-1}b=\sin^{-1}a$.

Thus we have $$S(a)=2\,g(b),$$

or $$a+\frac{2}{3^2}a^3+\frac{2\cdot4}{3\cdot5^2}a^5+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}a^7+\ldots=\beta\log b+2\sum_0^\infty\frac{\sin(2n+1)\beta}{(2n+1)^2},$$

a result analogous to that in Question 585.

When $\beta=\frac{\pi}{6}$, $a=\frac{1}{2}$, $b=(2-\sqrt{3})$; and we get

$$S(\tfrac{1}{2})=2g\,(2-\sqrt{3})$$

$$=\frac{\pi}{6}\log(2-\sqrt{3})+2\left\{\frac{1}{2}\cdot\frac{1}{1^2}+1.\frac{1}{3^2}+\frac{1}{2}\cdot\frac{1}{5^2}-\frac{1}{2}\cdot\frac{1}{7^2}-1.\frac{1}{9^2}-\frac{1}{2}\cdot\frac{1}{11^2}\ldots\right\}$$

$$=\frac{\pi}{6}\log(2-\sqrt{3})+\frac{4}{3}g(1),\text{ on reduction.}$$

Hence $$\frac{2}{3}S(1)-S(\tfrac{1}{2})=-\frac{\pi}{6}\log(2-\sqrt{3})$$

or $$\tfrac{1}{3}S(1)-\tfrac{1}{2}S(\tfrac{1}{2})=\frac{\pi}{12}\log(2+\sqrt{3}),$$

which gives Q. 546 (i) directly.

Putting $\beta=\frac{\pi}{4},\frac{\pi}{3}$ successively and simplifying, we shall get the following identities :

$$1+\frac{2}{3^2}(\tfrac{1}{2})+\frac{2\cdot4}{3\cdot5^2}(\tfrac{1}{2})^2+\ldots=-\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$
$$+2\left\{\frac{1}{1^2}+\frac{1}{3^2}-\frac{1}{5^2}-\frac{1}{7^2}+\ldots\right\},$$

$$1+\frac{2}{3^2}(\tfrac{3}{4})+\frac{2\cdot4}{3\cdot5^2}(\tfrac{3}{4})^2+\ldots=-\frac{\pi}{3\sqrt{3}}\log 3$$
$$+2\left\{\frac{1}{1^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{11^2}+\ldots\right\}.$$

K. J. SANJANA.

Geometrical Proof of a Trigonometrical Identity.

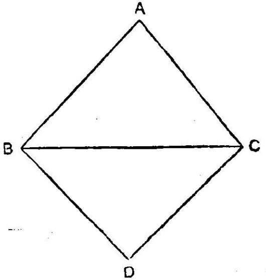
In the *Mathematical Notes* for January 1915 published by the Edinburgh Mathematical Society, Dr. R. J. T. Bell draws attention to a simple geometrical proof of the identity

$$1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C=0,^*$$

where A, B, C are the angles of a triangle.

The following proof appears to be even simpler than the one suggested by Dr. Bell.

Let ABC be any triangle. Draw BD, CD at right angles to AB, AC. Then in the triangle BCD



(i) $BD : CD : BC = \sin BCD : \sin CBD : \sin D$
$= \cos C : \cos B : \sin A.$

(ii) $BC^2 = BD^2 + CD^2 - 2\, BD.CD.\cos D$
$= BD^2 + CD^2 + 2BD \cdot CD \cdot \cos A.$

Hence from (i) and (ii)

$$\sin^2 A = \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C,$$

which gives the result.

M. T. NARANIENGAR.

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* See also *A New Trigonometry* by Lock and Child, p. 344.

SOLUTIONS.

Question 386.

(S. Ramanujan):—Shew that

$$\int_0^\infty \frac{dx}{(1+x^2)(1+n^2x^2)(1+n^6x^2)\ldots} = \frac{\pi}{2(1+n+n^3+n^6+n^{10}+\ldots)}$$

where the indices in the denominator are the sums of the natural numbers.

Solution by N. Durai Rajan.

Let $\quad \mathrm{I}=\int_0^\infty \frac{dx}{(1+x^2)(1+n^2x^2)(1+n^4x^2)(1+n^6x^2)}\cdots$

and write $\quad \frac{1}{(1+x^2)\ldots\text{to }\infty}\equiv\frac{a_0}{1+x^2}+\frac{a_1}{1+n^2x^2}+\ldots\frac{a_r}{1+n^{2r}x^2}+\ldots$

By putting $x^2=-n^{2r}$, we get

$$1/a_r=\left(1-\frac{1}{n^{2r}}\right)\left(1-\frac{n^2}{n^{2r}}\right)\ldots\ldots\left(1-\frac{n^{2r-2}}{n^{2r}}\right)\left(1-\frac{n^{2r+2}}{n^{2r}}\right)\ldots\ldots$$

$$=\left(1-\frac{1}{n^2}\right)\ldots\ldots\left(1-\frac{1}{n^{2r}}\right)(1-n^2)(1-n^4)(1-n^6)\ldots\text{to }\infty$$

$$=\left(1-\frac{1}{n^2}\right)\ldots\ldots\left(1-\frac{1}{n^{2r}}\right)\phi(n)\text{ say}$$

$$\therefore\ \mathrm{I}=\sum\int_0^\infty\frac{a_r dx}{1+n^{2r}x^2}=\sum\int_0^\infty\frac{a_r}{n^r}\cdot\frac{dy}{1+y^2}.\quad [n^r x=y]$$

$$=\frac{\pi}{2}\sum\frac{a_r}{n^r}.$$

But $\frac{a_r}{n^r}=\frac{1}{n^r}\cdot\frac{1}{\phi(n)}\cdot\frac{1}{\left(1-\frac{1}{n^2}\right)\ldots\left(1-\frac{1}{n^{2r}}\right)}=\frac{1}{n^r}\cdot\frac{1}{\phi(n)}\frac{n^2\cdot n^4\ldots n^{2r}}{(1-n^2)\ldots(1-n^{2r})}(-)^r$

$$=\frac{1}{\phi(n)}\frac{(-1)^r n^{r(r+1)}}{n^r\cdot(1-n^2)\ldots(1-n^{2r})}=\frac{1}{\phi(n)}\cdot\frac{n^{r^2}}{(1-n^2)\ldots(1-n^{2r})}(-)^r$$

$$\therefore\quad\sum\frac{a^r}{n^r}=\frac{1}{\phi(n)}\left[1-\frac{n}{1-n^2}+\frac{n^4}{(1-n^2)(1-n^4)}-\frac{n^9}{(1-n^2)(1-n^4)(1-n^6)}+\ldots\ldots\right]$$

$$=\frac{(1-n)(1-n^3)(1-n^5)\ldots\ldots\text{to }\infty}{(1-n^2)(1-n^4)(1-n^6)\ldots\ldots\text{to }\infty}:$$

(Chrystal, Part II, p. 344, Ex. 7.)

$$\therefore\quad \mathrm{I}=\frac{\pi}{2}\,\frac{(1-n)(1-n^3)(1-n^5)\ldots\ldots\text{to }\infty}{(1-n^2)(1-n^4)(1-n^6)\ldots\ldots\text{to }\infty}.$$

But by a theorem of Gauss (See Ex. 10 p. 345, Chrystal)

$$1+\sum x^{\frac{1}{2}n(n+1)}=\frac{(1-n^2)(1-n^4)(1-n^6)}{(1-n)(1-n^3)(1-n^5)}\dots\infty$$

$$\therefore \qquad I=\frac{\pi}{2(1+n+n^3+n^6+\dots)}$$

Question 557.

(Professor SANJANA) :—At a point P of a parabola the normal chord PQ is drawn meeting the axis in G and PN is the ordinate to the axis, prove that

$$PQ\cdot AN=2\ SP\cdot PG.$$

If from P two chords PP_1, PP_2 are drawn normal to the curve at P_1, P_2, prove further that

$$SP_1\cdot SP_2=AS.SP,\ PP_1\cdot PP_2=PQ\cdot AN$$

where S is the focus.

Shew also that the circle P_1P_2S passes through the circumcentre of PP_1P_2.

Note by the Proposer.

With regard to Mr. V. V. S. Narayan's solution of this Question in the Journal for April last, I have to point out that the first two properties were discovered long ago; the second was published in the *Educational Times* for June 1903, and has been solved in Vol. VI of the *E. T. Reprints* (New Series). The last part was discovered about four years ago.

Question 561.

(P. A. SUBRAMANIYA AIYAR, B.A., L.T.) :—Solve completely

$$(1+3x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-4y=0$$

Solution by A. Narasinga Rao.

Put $y=(2x^2+1)\int u dx$. The equation transforms into

$$\frac{du}{dx}+\frac{22x^3+7x}{(2x^2+1)(3x^2+1)}u=0.$$

i.e.,
$$\frac{du}{u}+\left(\frac{8}{2x^2+1}-\frac{1}{3x^2+1}\right)xdx=0.$$

$$\therefore \qquad u=\frac{A(3x^2+1)^{\frac{4}{3}}}{(2x^2+1)^2}.$$

$$\therefore \qquad y=A\int\frac{(3x^2+1)^{\frac{4}{3}}}{(2x^2+1)^2}dx+B(2x^2+1),$$

where A and B are arbitrary constants.

Question 585.

(K. V. ANANTHANARAYAN SASTRI, B. A.) :—Prove that

$$\int_0^{\frac{\pi}{2}} \tan^{-1}(a \sin \phi)\, d\phi = \frac{\pi^2}{4} - \log m \log \frac{1-m}{1+m}$$
$$-2\sum_0^{\infty} \frac{m^{2r+1}}{(2r+1)^2},$$

where $m=(1+a^2)^{\frac{1}{2}}-a.$

Solution by T. P. Trivedi M.A., LL.B., and K. B. Madhava.

We have

$$2\sum \frac{m^{2r+1}}{(2m+1)^2} = \int_0^m \frac{1}{x}\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right)dx$$

$$= \int_0^m \frac{\tanh^{-1}x}{x}dx = 2\Big[\tanh^{-1}x \ \log x\Big]_0^m - 2\int_0^m \frac{\log x\, dx}{1-x^2}$$

$$= \log m \log \frac{1+m}{1-m} - 2\int_1^m \frac{\log x\, dx}{1-x^2} - 2\int_0^1 \frac{\log x\, dx}{1-x^2}$$

$$= \log m \log \frac{1+m}{1-m} - 2\int_0^n \frac{\log \frac{1-\zeta}{1+\zeta}\, d\zeta}{1-\frac{(1-\zeta)^2}{(1+\zeta)^2}} \times -\frac{2}{(1+\zeta)^2} + \frac{\pi^2}{4}$$

$$\left(\text{where in the second integral } n=\frac{1-m}{1+m},\ x=\frac{1-\zeta}{1+\zeta}\right),$$

$$= \log m \log \frac{1+m}{1-m} - 2\int_0^n \frac{\tanh^{-1}\zeta}{\zeta}d\zeta + \frac{\pi^2}{4}$$

$$= \log m \log \frac{1+m}{1-m} - \int_0^n \frac{2\tanh^{-1}\zeta}{\zeta}d\zeta + \frac{\pi^2}{4}$$

$$= \log m \log \frac{1+m}{1-m} - \int_0^a \frac{2\tanh^{-1}\left(\frac{\sqrt{1+y^2}-1}{y}\right)}{y\sqrt{1+y^2}}dy + \frac{\pi^2}{4}$$

$$\text{if } \zeta = \frac{\sqrt{1+y^2}-1}{y}$$

$$= \log m \log \frac{1+m}{1-m} - \int_0^a \frac{\tanh^{-1}y}{y\sqrt{1+y^2}}dy + \frac{\pi^2}{4}$$

$$=\log m \log \frac{1+m}{1-m}+a-\frac{a^3}{3}\cdot\frac{2}{3}+\frac{a^5}{5}\cdot\frac{2\cdot4}{3\cdot5}-\frac{a^7}{7}\cdot\frac{2\cdot4\cdot6}{3\cdot5\cdot7}+\dots+\frac{\pi^2}{4}$$

$$=\log m \log \frac{1+m}{1-m}+\int_0^{\frac{\pi}{2}}\left\{a \sin\phi-\frac{a^3}{3}\sin^3\phi+\dots\right\}d\phi+\frac{\pi^2}{4}$$

$$=\log m \log \frac{1+m}{1-m}+\int_0^{\frac{\pi}{2}}\tan^{-1}(a\sin\phi)\,d\phi+\frac{\pi^2}{4}.$$

Hence the result.

Question 587.

(T. P. TRIVEDI, M. A., L.L.B.) :—If the centre of pressure of a triangle immersed in a liquid coincides with the symmedian point prove that the depths of the angular points are in the ratios

$$3a^2-b^2-c^2 : 3b^2-c^2-a^2 : 3c^2-a^2-b^2.$$

Solution by D. Krishnamurti, M. K. Veeraraghavan and K. B. Madhava.

This question is only a particular case of the following :—

A triangular lamina is totally immersed in a homogeneous liquid the depths of the angular points being p, q, r : shew that if the centre of pressure of the triangle coincide with the mean centre of its angular points for multiples l, m, n, then $p : q : r = 3l-m-n$; $3m-n-l$; $3n-l-m$.

[Besant and Ramsey : *Hydro-Mechanics*, Part I, p. 58, Q. 46.]

We can easily show that the areal co-ordinates of the centre of mass for multiples l, m, n at the angular points of a triangle are proportional to l, m, n.

Now the centre of pressure P of ABC is the centroid of masses $q+r, r+p, p+q$ at A′, B′, C′ the mid-points of the sides : [*Vide*: Greaves : *Hydrostatics*, article 47.]

Let $(x'y'z')$ and (xyz) be the co-ordinates of the centre of pressure referred to $\triangle^s$ A′B′C′ and ABC. Then from the above property of the centre of mass, we see that

$$x' : y' : z' = q+r : r+p : p+q ;$$

i.e.

$$y+z-x : z+x-y : x+y-z = q+r : r+p : p+q.$$

From this we easily get

$$p : q : r = 3x-(y+z) : 3y-(z+x) : 3z-(x+y).$$

i.e,

$$p : q : r = 3l-m-n : 3m-n-l : 3n-l-m.$$

Since the symmedian point is the mean centre for masses a^2, b^2, c^2 at A,B,C,

$$l : m : n = a^2 : b^2 : c^2.$$

$$\therefore \quad p : q : r = 3a^2 - b^2 - c^2 : 3b^2 - c^2 - a^2 : 3c^2 - a^2 - b^2.$$

Question 589.

(S. KRISHNASWAMI AIYANGAR) :—ABCD is a quadrilateral and its diagonals meet at right angles at S. With S as focus two conics α, β are described one circumscribing the quadrilateral and the other inscribed in it. Shew that

(1) the two conics have one directrix identical;

(2) the envelope of the polars with respect to β of points on α is a conic having one focus and one directrix common with those of α, β;

(3) the lines joining the points of contact of β with the sides of the quadrilateral meet two by two at the vertices of the harmonic triangle of the quadrilateral.

Solution by L. N. Subramaniam, M.A., L. T.

Reciprocate α, β with respect to their common focus S. The quadrilateral ABCD reciprocates into a rectangle *abcd*, since the diagonals of ABCD intersect at right angles at the centre of reciprocation. Also α, β reciprocate into α', β' respectively inscribed and circumscribed in the rectangle *abcd*, β' being a circle; so that, β' is the director circle of the conic α' and is concentric with it.

Now, the common centre of α', β' corresponds to the S-directrix of α, β and therefore we see that the conics α, β have a common directrix; which is result (1)

Since the locus of the pole of a tangent to α' with respect to β' is evidently a concentric ellipse, we have the reciprocal theorem (2), *viz*: the envelope of the polars with respect to β of points on α is a conic with S as focus, and the same directrix as α and β.

Lastly, since the rectangle *abcd* is inscribed in β', result (3) follows at once by reciprocation. It is, in fact, the well-known harmonic property of a circumscribed quadrilateral.

Question 592.

(S. NARAYANA AIYAR, M.A.,) :—Show that a^n is equal to

$$\begin{vmatrix} 1, & 0, & 0, & \dots & 0, & 0 & 1 \\ 1, & 1, & 0, & \dots & 0 & 0 & 1+a \\ 1, & 2, & 1, & \dots & 0 & 0 & (1+a)^2 \\ 1, & {}_{n-1}C_1 & {}_{n-1}C_2 & \dots & n-1, & 1, & (1+a)^{n-1} \\ 1 & {}_nC_1 & {}_nC_2 & \dots & {}_nC_2 & {}_nC_1 & (1+a)^n. \end{vmatrix}$$

Solution (1) by T. P. Trivedi, M.A., LL.B., D. Krishnamurti, K. B. Madhava and N. Sankara Aiyar, M.A.; (2) by A. Narasinga Rao and N. B. Pendse.

(1) Denote the determinant by D_{n+1}; subtracting each row from the next,

$$D_{n+1}=\begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & a \\ 0 & 1 & 1 & \dots & 0, & a(1+a) \\ 0 & 1 & {}_{n-2}C_1 & \dots & 1 & a(1+a)^{n-2} \\ 0 & 1 & {}_{n-1}C_1 & \dots & {}_{n-1}C_1 & a(1+a)^{n-1} \end{vmatrix}$$

which is clearly the same as the determinant formed by deleting the first column and row; thus

$$D_{n+1}=a\cdot D_n.$$

By the repeated application of the above, we get

$$D_{n+1}=a^{n-1}.D_2$$

But $$D_2=\begin{vmatrix} 1, & 1 \\ 1, & 1+a \end{vmatrix}=a$$

Hence the result.

(2) Eliminate $a^0, a^1, a^2, \dots a^{n-1}$ from the $(n+1)$ identities

$$1 \quad =1$$
$$1+a \quad =1+ a$$
$$(1+a)^2=1+2a+a^2$$
$$(1+a)^n-a^n=1+n^c{}_1\, a+n^c{}_2\, a^2+\dots\dots\dots+na^{n-1}.$$

Then $D-a^nD'=0$, where D is the given determinant and D′ the minor of the last constituent of its leading diagonal.

But obviously $D'=1$.

Hence $D=a^n$.

Question 593.

(S. Narayana Aiyar, M.A.,) :—Show that,

$$\frac{\frac{1}{a}+\frac{x}{a+1}+\frac{x^2}{1\cdot2}\cdot\frac{1}{a+2}+\frac{x^3}{1\cdot2\cdot3}\cdot\frac{1}{a+3}+\ldots\text{to }\infty}{\frac{1}{a+1}+\frac{x}{a+2}+\frac{x^2}{1\cdot2}\cdot\frac{1}{a+3}+\frac{x^3}{1\cdot2\cdot3}\cdot\frac{1}{a+4}+\ldots\text{to }\infty}\quad\text{is equal to}$$

$$\frac{\frac{1}{a}-\frac{x}{a(a+1)}+\frac{x^2}{a(a+1)(a+2)}-\frac{x^3}{a(a+1)(a+2)(a+3)}+\ldots\text{to }\infty}{\frac{1}{a+1}-\frac{x}{(a+1)(a+2)}+\frac{x^2}{(a+1)(a+2)(a+3)}-\ldots\text{ to }\infty.}$$

Solution (1) *by. K.B. Madhava* ; (2) *by T.P. Trivedi M.A., LL.B., M. Bhimasena Rao and K. Appukuttan Erady, M.A.*

(1) We have by the method of partial fractions

$$\frac{x^n}{a(a+1)\ldots(a+n)}=\frac{x^n}{a}\cdot\frac{1}{n!}-\frac{x^n}{(a+1)\,1!}\cdot\frac{1}{n-1!}+$$

$$\frac{x^n}{(a+2)\,2!}\cdot\frac{1}{n-2!}-\ldots+\frac{1}{n!}\frac{x^n}{a+n}$$

from which we have successively

$$\frac{1}{a}=\frac{1}{a}$$

$$-\frac{x}{a(a+1)}=-\frac{x}{1!}\cdot\frac{1}{a}+\frac{x}{1!\,(a+1)}$$

$$\frac{x^2}{a(a+1)(a+2)}=\frac{x^2}{2!}\cdot\frac{1}{a}-\frac{x}{1!}\cdot\frac{x}{1!\,(a+1)}+\frac{x^2}{2!\,(a+2)}$$

and so on ; so that

$$\frac{1}{a}-\frac{x}{a(a+1)}+\frac{x^2}{a(a+1)(a+2)}-\frac{x^3}{a(a+1)(a+2)(a+3)}+\ldots\text{to }\infty$$

$$=e^{-x}\left[\frac{1}{a}+\frac{x}{a+1}+\frac{x^2}{2!}\cdot\frac{1}{a+2}+\frac{x^3}{3!}\cdot\frac{1}{a+3}+\ldots\text{to }\infty\right].$$

In the same way, substituting $a+1$ for a throughout, we see that the denominator of the right hand expression is e^{-x} times the denominator of the left, from which the given equality follows at once.

(2) The left side

$$=\frac{\frac{1}{x^a}\int_0^x\left(x^{a-1}+\frac{x^a}{\underline{|1}}+\frac{x^{a+1}}{\underline{|2}}+\ldots\text{to }\infty\right)dx}{\frac{1}{x^{a+1}}\int_0^x\left(x^a+\frac{x^{a+1}}{\underline{|1}}+\frac{x^{a+2}}{\underline{|2}}+\ldots\text{to }\infty\right)dx}=\frac{x\int_0^x e^x.x^{a-1}\,dx}{\int_0^x e^x.x^a\,dx}\ ;$$

$$\frac{x\left\{\frac{x^{a}}{a}e^{x}-\frac{x^{a+1}}{a(a+1)}e^{x}+\frac{x^{a+2}}{a(a+1)(a+2)}e^{x}-\dots\right\}}{\left\{\frac{x^{a+1}}{a+1}e^{x}-\frac{x^{a+2}}{(a+1)(a+2)}e^{x}+\frac{x^{a+3}}{(a+1)(a+2)(a+3)}e^{x}-\dots\right\}}$$

$$=\frac{\frac{1}{a}-\frac{x}{a(a+1)}+\frac{x^2}{a(a+1)(a+2)}-\dots\text{to }\infty}{\frac{1}{a+1}-\frac{x}{(a+1)(a+2)}+\frac{x^2}{(a+1)(a+2)(a+3)}-\dots\text{to }\infty}$$

on applying the process of integration by parts continuously.

Question 594.

(A. A. KRISHNASAWAMI AIYANGAR, B. A.) :—Establish the following approximate formula for the length of a circular arc

$$\frac{32768R-5376Q+168H-C}{2835}.$$

where C is the chord of the whole arc,
H........................half the arc,
Q........................quarter of the arc,
R........................eighth of the arc.

Remarks by K. B. Madhava, T. P. Trivedi, R. J. Pocock, D. Krishnamurti, N. B. Pendse and S. V. Venkataraya Sastri, M.A., L.T.

The denominator was wrongly printed 2735.

Using Mr. R. Vythynathaswamy's "*Note on the length of the Circular Arc*" in Volume VI, No. 4 of the *J. I. M. S.*, and with the same notation,

$$s=\theta=a_1\sin p\theta+a_2\sin p^2\theta+a_3\sin p^3\theta+a_4\sin p^4\theta.$$

If $p=\frac{1}{2}$, $n=4$, we get

$$a_1=-\frac{2}{2835},\ a_2=\frac{16}{135},\ a_3=-\frac{512}{135},\text{ and }a_4=\frac{65536}{2835}.$$

Hence the required result.

Question 596.

(N. SANKARA IYER, M.A.) :—Show that

$$\frac{1}{2}\frac{1}{2}-\frac{1\cdot3}{2\cdot4^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6^2}-\dots=\log\frac{1+\sqrt{2}}{2},$$

and

$$\frac{1}{2}\frac{1}{2}+\frac{1\cdot3}{2\cdot4^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6^2}+\dots=\log 2.$$

Solution by K. B. Madhava, D. Krishnamurti, T. P. Trivedi and N. B. Pendse.

Expand $\log \frac{1}{2}\{1+\sqrt{1+x}\}$ by Lagrange's Series, or more easily by the differential equation $\frac{dy}{dx}=-\frac{1}{2a}\left\{(1+x)^{-\frac{1}{2}}-1\right\}x$ in the form

$$\frac{1}{2}\frac{x}{2}-\frac{1\cdot 3}{2\cdot 4}\frac{x^2}{4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^3}{6}-\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\frac{x^4}{8}\cdots\cdots$$

and put $x=1$ to obtain the first result. The validity of such a step which is not expressly sanctioned by the expansion of $(1+x)^{-\frac{1}{2}}$ for $x=1$ may be questioned. This is permitted by the following theorem which I quote from Bromwich's *Infinite Series*, (P. 204)

"As a special example we may take the hyper-geometric series in which $\frac{a_n}{a_{n+1}}=\frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)}$, so that $\mu=1+\gamma-(\alpha+\beta)$.

Thus we see that for $x=1$, (i) if the real part of $(\gamma-\alpha-\beta)$ is positive, the hypergeometric series converges absolutely, etc."

Here $\gamma=1$, $\alpha=0$, $\beta=\frac{1}{2}$ and $\gamma-\alpha-\beta=\frac{1}{2}$, which is positive.

For the second also the same remarks apply:

Since $$(1-x^2)^{-\frac{1}{2}}=1+\frac{1}{2}x^2+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2}x^4\cdots\cdots$$

$$\int\frac{(1-x^2)^{-\frac{1}{2}}-1}{x}dx=\frac{1}{2}\frac{x^2}{2}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2}\cdot\frac{x^4}{4}+\frac{\frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}}{1\cdot 2\cdot 3}\frac{x^6}{6}\cdots\cdots$$

which gives

$$\frac{1}{2}\cdot\frac{1}{2}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2}\frac{1}{4}+\cdots\cdots\cdots=\int_0^{\frac{\pi}{2}}\frac{1-\cos\theta}{\sin\theta}d\theta$$

$$=-\left[\log\cos^2\frac{\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$=\log 2.$$

In fact $$\frac{a_n}{a_{n+1}}=\frac{(2n+2)(2n+2)}{(2n)(2n+1)}.$$

Question 597.

(A. Narasinga Rao) :—Show that the parabola is the only curve such that the triangle formed by any three points on the curve is proportional to the triangle formed by the tangents at these points.

Solution by N. Durai Rajan and N. B. Pendse.

Take a tangent and normal to the curve as axes of x and y. Let P, P' be two adjacent points on the curve and PT, P'T' the tangents meeting the x—axis in T, T' and intersecting in Q.

By hypothesis

$\Delta OPP' = \Delta QTT' \times k.$

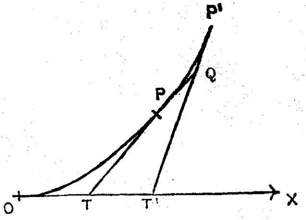
In the limit

$\Delta OPP' = \Delta PTT' \times k.$

$\therefore \ \frac{1}{2} r^2 d\theta = \frac{1}{2} k PT^2 . d\psi.$

$$\therefore \ r^2 d\theta = \frac{ky^2}{\sin^2\psi} . d\psi$$

$$= \frac{kr^2 \sin^2\theta}{\sin^2\psi} . d\psi.$$



Thus
$$\frac{d\theta}{\sin^2\theta} = k \frac{d\psi}{\sin^2\psi}.$$

Integrating
$$\cot\theta = A + k\cot\psi.$$

$$\therefore \qquad \frac{x}{y} = A + k\frac{dx}{dy}.$$

This is a homogeneous differential equation; by putting $x = vy$, we get for the primitive

$$Ay + x(k-1) = By^k.$$

On changing the constants

$$(x + \alpha y)^k = \beta y \qquad \ldots \qquad \ldots \qquad \ldots (1)$$

Take $x + \alpha y = 0$ and $y = 0$ as the new axes of y & x. The equation of the curve with then be of the form $y = a x^k$.

Let $(x_1 y_1)$ $(x_2 y_2)$ be two points on the curve; the tangents at these points are

$$Y - ax_1^k = a_1 \cdot k \cdot x^{k-1}(X - x_1)\text{ ; } etc.$$

$$\therefore \qquad \frac{Y}{ax_1^{k-1}} = X \cdot k - (k-1)x_1. \qquad \ldots \qquad \ldots \qquad \ldots (2)$$

Similarly
$$\frac{Y}{ax_2^{k-1}} = X \cdot k - (k-1)x_2. \qquad \ldots \qquad \ldots \qquad \ldots (3)$$

The ordinate of intersection is given by

$$\frac{Y}{a}\left(x_2^{k-1} - x_1^{k-1}\right) = (x_2 x_1)^{k-1}(k-1)(x_2 - x_1).$$

The intercepted part of the x—axis between (1) and (2) is

$$\frac{k-1}{k}(x_2 - x_1).$$

By hypothesis, $\Delta OP_1P_2 = \Delta QT_1T_2 \cdot k$

$\therefore \quad \Delta O P_1 P_2 = \frac{1}{2} \sin w \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \frac{1}{2} \sin w (x_2 x_1)(x_2^{k-1} - x_1^{k-1}) a$

$$= k \cdot \Delta Q T_1 T_2 = \frac{1}{2} \, Y \sin w \frac{k-1}{k} (x_2 - x_1) \times k$$

$$= \frac{1}{2} \cdot \frac{\sin w . \; a (x_2 x_1)^{k-1} (k-1)(x_2 - x_1)^2}{(x_2^{k-1} - x_1^{k-1})} \cdot \frac{k-1}{k} \times k$$

$\therefore \quad (x_2^{k-1} - x_1^{k-1})^2 = (x_2 x_1)^{k-2} (x_2 - x_1)^2 (k-1)^2.$

This equation must be true for all values of x_1 and x_2.

$\therefore \qquad k = 2.$

Hence the curve is a parabola.

Remarks. A similar problem is given in Forsyth's *Differential Equations*, Ex. 9., P. 49, 3rd Edition. The problem is: " The tangent at any point P of a curve cuts the tangent and normal at a fixed point O in the points M and N and the rectangle OMP′N is completed. Find the curve which is such that the triangle formed by the tangents at any three points P, Q, R is equal to the Δ formed by the corresponding points P′ Q′ R′."

A different method of attacking this has to be adopted.

Since the theorem is true for any three points P, Q, R, it is also true when one of the points is the origin.

Let $\qquad \frac{x}{a} + \frac{y}{b} = 1, \; \frac{x}{a'} + \frac{y}{b'} = 1$

be the tangents at Q and R. The points Q′, R′ are (a, b) and (a', b').

$$\Delta \text{ of tangents} = \frac{1}{2} (a - a') \frac{bb'}{ab' - a'b}$$

$$= \Delta \; O \; Q' \; R'$$

$$= \frac{1}{2} (ab' - a'b)$$

$\therefore \qquad bb' (a - a')^2 = (ab' - a'b)^2 \qquad \ldots \qquad \ldots \qquad \ldots$ (1)

This is true for all values of a, a', while a and b, a' and b' are connected by the same fixed relation.

From (1) $\qquad bb' (a^2 + a'^2 - 2aa') = a^2 b'^2 + a'^2 b^2 - 2aa' \, bb'$

$\therefore \qquad a^2 (b - b') b' = a'^2 (b - b') b$

$\therefore \qquad \frac{a^2}{b} = \frac{a'^2}{b'}.$

Hence we see that

$$\frac{a^2}{b} = \frac{a'^2}{b'} = \frac{a''^2}{b''} = \ldots = \text{constant, } \lambda \text{ say.}$$

$\therefore \qquad a^2 = b \cdot \lambda.$

The equation of the tangent is

$$\frac{x}{a}+\frac{\lambda y}{a^2}=1$$

or

$$a^2-ax-\lambda y=0.$$

The equation of the envelope is

$$x^2+4\lambda y=0$$

which is a parabola.

Also since

$$\Delta(123)=\Delta(012)+\Delta(023)+\Delta(031)$$

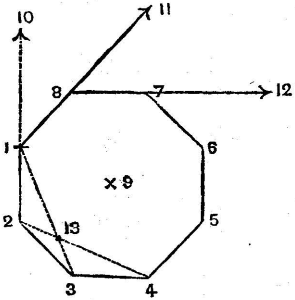
where 0, 1, 2, 3 denote either tangents or points, due regard being had to signs of areas, we see that the parabola satisfies the proposed condition generally.

Question 598.

(A. NARASINGA RAO) :—Arrange 13 points so as to form 18 rows of 3 each, no row containing more than 3.

Solution (1) *by N. Durai Rajan and* (2) *by N. B. Pandse.*

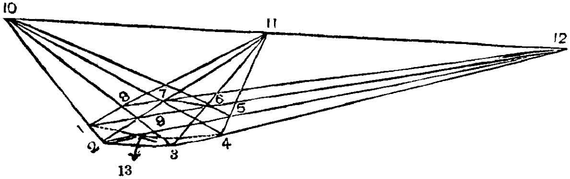
Adopting a hint given in Ball's *Mathematical Recreations*, we may try the conical projection of a regular octagon.



Place one point at each of the 8 angular points,
one at a point at infinity on line 12 (point 10)
one „ „ 18 („ 11)
one „ „ 87 („ 12)
one at the centre (point 9).
one at the intersection of lines 13 and 24 (point 13).

Project the regular octagon from any vertex.

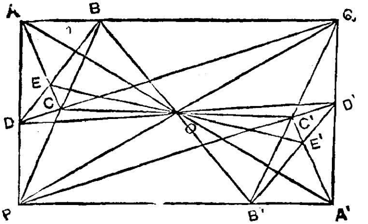
In the projected figure we have 13 points of which there are
4 rows (2, 1, 10 ; 3, 8, 10 ; 4, 7, 10 ; 5, 6, 10)
4 rows (1, 8, 11 ; 2, 7, 11 ; . . .)
4 rows (8, 7, 12 ; . . . . .)
4 rows (1, 9, 5 ; 2, 9, 6 ; 3, 9, 7 ; 4, 9, 8 ;)
1 row (10, 11, 12)
2 rows (1, 13, 3 ; 2, 13, 4.)
1 row (13, 9, 11).
since the join of 9, 13 in the regular octagon is parallel to the side (1, 8).



Thus we have 20 rows of 3 each.

One method of obtaining the projected figure is to draw the perspective representation of a horizontal regular octagon, in which points 10, 11, 12 are points on the vanishing line.

If we place 13 at any other point of line (13) or (24) we get only 18 rows.



(2) Take any two points P, Q and let their middle point be O. Draw two straight lines from P and two from Q to cut in 4 points A, B, C, D. From Q draw straight lines parallel to those at P and from P parallel to those at Q, cutting in A′ B′ C′ D′. Join A C and B D cutting at E, and A′ C′ and B′ D′ cutting at E′. It can now be shown that AA′, BB′, CC′, DD′ and EE′ are all bisected at O.

Thus there are 13 points and 18 straight lines such that not more than three are in a straight line.

Question 599.

(V. V. S. Narayan) :—If two circles cut orthogonally, find the locus of a point such that the tangents from it to the two circles form a harmonic pencil.

Solution by S. V. Venkataraya Sastri, M.A., L.T.

Let the circles be $x^2+y^2+2gx-a^2=0$, and $x^2+y^2+2fx-a^2=0$.

Since they cut orthogonally, we have $fg=a^2$.

The equation of the tangents from (x_1, y_1) to the first circle is

$$(x^2+y^2+2gx-a^2)(x_1^2+y_1^2+2gx_1-a^2)=\{xx_1+yy_1+g(x+x_1)-a^2\}^2$$

i.e., $$x^2(y_1^2-a^2-g^2)-2xyy_1(x_1+g)+y^2(x_1^2+2gx_1-a^2)+\ldots=0.$$

Similarly, the equation of the tangents to the second circle is

$$x^2(y_1^2-a^2-f^2)-2xyy_1(x_1+f)+y^2(x_1^2+2fx_1-a^2)+\ldots=0.$$

These four lines will form a harmonic pencil, if

$$(y_1^2-a^2-g^2)(x_1^2+2fx_1-a^2)+(x_1^2+2gx_1-a^2)(y_1^2-a^2-f^2)$$
$$=2y_1^2(x_1+g)(x_1+f),$$

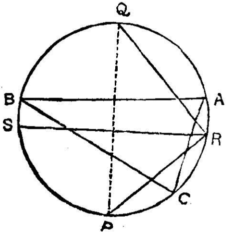
which, when simplified, gives $x_1^2-a^2=0$.

Hence the required locus consists of two straight lines parallel to the common chord of the given circles and at a distance from it equal to half the common chord.

Question 600.

(V. V. S. Narayan) :—If PQ is a diameter of the circum-circle of a triangle ABC, shew that the axes of the inscribed parabolas whose foci are P and Q intersect at a point R on the circle. Shew further that the axis of the inscribed parabola whose focus is R is perpendicular to PQ.

Solution by D. Krishnamurti and N. B. Pendse.



If any point P is taken and a parabola is drawn with P as focus to touch the sides of the triangle, then the axis and AP are equally inclined to B C.

Hence the axes corresponding to two points make the same angle which the points subtend at A.

Since PQ is a diameter PAQ is a right angle. Therefore the axes are at right angles and meet on the circle.

Let them meet at R. The axis corresponding to R makes with PR an angle equal to the angle PAR, RS being that axis.

$$\angle PRS = \angle PAR.$$

$$\therefore \quad \text{arc } PS = \text{arc } PR.$$

Thus RS is at right angles to PQ.

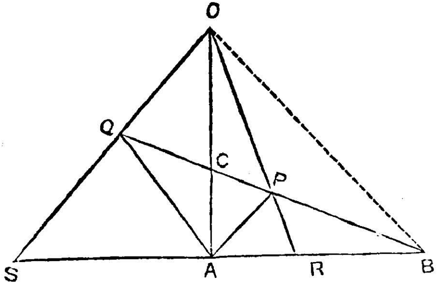
Question 602.

(S. Krishnaswami Iyengar) :—A variable straight line meets two fixed straight lines intersecting at O in the points P, Q. If OP, OQ subtend equal angles at a fixed point A, shew that PQ passes through a fixed point B such that BAO is a right angle.

Solution by V. V. S. Narayan, R. D. Karve, S. V. Venkataraya Sastri, M.A., L.T. and several others.

Let the $\perp^r$ to OA through A cut PQ, OP, OQ at B, R, S. Let PQ cut OA at C. Join OB.

AC, AB are the internal and external bisectors of the angle PAQ since, by hypothesis, OP, OQ subtend equal angles at A, and since, by construction, $C\hat{A}B$ is a right angle.



$\therefore$ {BPCQ} is a harmonic range.

$\therefore$ O{BPCQ} *i.e.*, O{BRAS} is a harmonic pencil.

$\therefore$ {BRAS} is a harmonic range, and since R, A, S, are fixed by construction, B is a fixed point.

QUESTIONS FOR SOLUTION.

668. (J. C. SWAMINARAYAN, M.A.) :—If r and n are integers, prove that the expression

$$1-\frac{2n+1}{1}{}_rC_1+\frac{(2n+1)(2n+3)}{1\cdot3}{}_rC_2-\frac{(2n+1)(2n+3)(2n+5)}{1\cdot3\cdot5}{}_rC_3+$$
$$\ldots\ldots+(-1)^r\frac{(2n+1)(2n+3)\ldots\ldots(2n+2r-1)}{1\cdot3\cdot5\cdot\ldots\ldots(2r-1)}{}_rC_r$$

is equal to $\frac{(-2)^r\ nPr}{1\cdot3\cdot5\ldots\ldots(2r-1)}$, as long as r is not greater than n, but vanishes if $r>n$.

669. (J. C. SWAMINARAYAN, M. A.) :—Prove that when n is a positive integer

$$(b^2-a^2)^n+(2n+1)\frac{2n}{\underline{|2}}a^2(b^2-a^2)^{n-1}+$$
$$(2n+1)(2n+3)\frac{2n(2n-2)}{\underline{|4}}a^4(b^2-a^2)^{n-2}+\ldots\ldots\ldots+$$
$$\equiv b^{2n}+\frac{(2n)^2}{\underline{|2}}b^{2n-2}a^2+\frac{(2n)^2(2n-2)^2}{\underline{|4}}b^{2n-4}a^4+\ldots\ldots\ldots$$
$$+\frac{(2n)^2(2n-2)^2(2n-4)^2\ldots\ldots\ldots2^2}{\underline{|2n}}a^{2n}.$$

670. (K. J. SANJANA, M.A.) :—Prove that

$$\frac{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots}{\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots}=1+\frac{1}{\underline{|3}}\pi^2\frac{B_1}{2}+\frac{7}{\underline{|5}}\pi^4\frac{B_3}{2^3}+\frac{31}{\underline{|7}}\pi^6\frac{B_5}{2^5}+\ldots,$$

where B_1, B_3,...are the numbers of Bernoulli.

671. (K. J. SANJANA, M.A.) :—The numbers from 1 to 2^n-1 being arranged on n cards on which the least numbers put down are respectively 2^0, 2^1, 2^2,...2^{n-1}, as in Question 640, prove that the sum of the numbers on the rth card is $2^{2n-2}+2^{n+r-3}-2^{n-2}$.

672. (S. KRISHNASWAMI AIYANGAR) :—The vertex A of the triangle of reference lies without, on, or within the director circle of the maximum inscribed ellipse according as the angle A is acute, right, or obtuse.

673. (S. KRISHNASWAMI AIYANGAR) :—If the sides of a polygon of n sides subtend equal angles at a point S, then with S as focus two conics can be described, one circumscribing the polygon and the other

inscribed in it. Show that the envelope of polars with respect to the inscribed conic of points on the circum-conic is a conic having one focus and one directrix coincident with those of the original conics.

674. (M. BHIMASENA RAO) :—Evaluate

$$\int_0^\infty \left\{ \frac{\cos px \cdot \sin qx}{\cosh rx . \cos^{\prime} qx} \cdot \frac{dx}{x} \right\}$$

675. (K. APPUKUTTAN ERADY) :—Shew that

$$\text{(i)} \int_0^\pi \frac{e^{b \cos \theta} \cos (b \sin \theta)}{(1-2a \cos \theta+a^2)} d\theta = \frac{\pi e^{ab}}{1-a^2},$$

$$\text{(ii)} \int_0^\pi \frac{\log (1-2b \cos \theta+b^2)}{(1-2a \cos \theta+a^2)} d\theta = \frac{2\pi}{1-a^2} \log (1-ab).$$

676. (N. P. PANDYA) :—Solve

$$\frac{dy^4}{dx^4} + \left(\frac{dy}{dx}\right)^4 = xy.$$

677. (D. KRISHNAMURTI) :—A chain of length l and mass M is coiled at the edge of a table. A particle of mass m is fastened to one end of the chain and the other end is gently let slip over the edge of the table. Shew that the velocity of the particle immediately after it leaves the table is

$$k\sqrt{[2gl\ (1+k^2)/3]}$$

where $k = \mathrm{M}/(\mathrm{M}+m)$.

678. (D. KRISHNAMURTI) :—Shew that the centroids of parallelograms inscribed in a skew quadrilateral, so as to have their sides parallel to the diagonals of the quadrilateral, all lie in a straight line which is also the locus of the centres of the conicoids having the sides of the quadrilateral for generators.

679. (MARTYN M. THOMAS) :—At each point P of the ellipse $x^2/a^2+y^2/b^2=1$ is described an ellipse having three-pointic contact with it and having a focus at the origin. Shew that the locus of the second focus is

$$\left\{ (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} \right\}^{\frac{1}{2}} \left\{ (x/a)^{\frac{2}{3}}(b^2/2-a^2)+(y/b)^{\frac{2}{3}}(a^2/2-b^2) \right\} = (a^2-b^2).$$

[M. A. 1900.]

680. (R. VYTHYNATHASWAMY) :—If $l, m, n, \ldots, \lambda$ are all positive integers, find the greatest value of

$$lx+my+nz+\ldots\ldots\ldots$$

subject to the condition

$$x+y+z+\ldots\ldots = \lambda,$$

$x, y, z \ldots$ being zero or positive integers.

681. (S. Ramanujan) :—Solve in integers $x^3+y^3+z^3=1$, and deduce the following:

$$6^3+8^3=9^3-1,\ 9^3+10^3=12^3+1,\ 135^3+138^3=172^3-1,$$
$$791^3+812^3=1010^3-1,\ 11161^3+11468^3=14258^3+1,$$

and
$$65601^3+67402^3=83802^3+1.$$

682. (S. Ramanujan) :—Shew how to find the cube root of surds of the form $A+\sqrt[3]{B}$; and deduce that

$$\sqrt[3]{(\sqrt[3]{2}-1)}=\sqrt[3]{\left(\tfrac{1}{9}\right)}-\sqrt[3]{\left(\tfrac{2}{9}\right)}-\sqrt[3]{\left(\tfrac{4}{9}\right)}.$$

683. (Ganpatram R. Jani) :—If S_r is put for $1^r+2^r+3^r\ldots+n^r$, prove that

$$\begin{vmatrix} S_0, & S_1, & S_2, & \ldots & \ldots & S_{n-1} \\ S_1, & S_2, & S_2, & \ldots & \ldots & S_n \\ S_2, & S_3, & S_4, & \ldots & \ldots & S_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ S_{n-1}, & S_n, & S_{n+1}, & \ldots & \ldots & S_{2n-2} \end{vmatrix}=\left\{\frac{(n!)^n}{1^1.2^2.3^3\ldots\ldots n^n}\right\}^2.$$

The order of the determinant is given by the number of the natural numbers taken.

684. (M. Bhimasena Rao) :—Shew that

$$\sum_{n=1}^{n=\infty}\tan^{-1}\left\{\frac{\sin(2n\pi\sqrt{1-x^2}).\ e^{-2n\pi x}}{1-\cos(2n\pi\sqrt{1-x^2})\ e^{-2n\pi x}}\right\}$$
$$=\frac{\pi}{12}\sqrt{1-x^2}-\frac{1}{4}\cos^{-1}x;$$

and obtain the result

$$\tan^{-1}e^{-h\pi}-\tan^{-1}e^{-3h\pi}+\tan^{-1}e^{-5h\pi}-\ldots\ldots$$
$$=\frac{1}{4}\sin^{-1}\frac{\sqrt{13}-3}{8}\ .+\frac{1}{8}\sin^{-1}\frac{\sqrt{13}-3}{4},\text{ where } h=\sqrt{13/12}.$$

685. (S. Malhari Rao):—If n and p are any two positive integers, shew that the fraction $1/n$ may be expressed as the sum of a series of p different fractions, the numerator of each of which is 1 and the denominator a positive integer. If the least of the fractions be $\frac{(n-1)!}{(n+p-1)!}$, find the rth fraction.